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Fractional Fourier series Expansion of Two Types of Fractional Trigonometric Functions

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Abstract: **In this paper, we find the fractional Fourier series expansion of two type of fractional trigonometric functions based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus. A new multiplication of fractional analytic functions plays an important role in this paper. The main methods we used are fractional Euler's formula and the fractional power series expansion of complex fractional analytic function. On the other hand, two examples are provided to illustrate our results.**

Keywords: **fractional Fourier series expansion, fractional trigonometric functions, Jumarie's modified R-L fractional calculus, new multiplication, fractional Euler's formula, complex fractional analytic function.**

I. INTRODUCTION

Fractional calculus originated in 1695, almost at the same time as traditional calculus. However, despite the initial contributions of important mathematicians, physicists and engineers, fractional calculus has attracted limited attention and is still a pure mathematical exercise. Fractional calculus has developed rapidly in mathematics and applied science in the past few decades. Now it is considered as an excellent tool to describe complex systems, involving long range memory effects and non local phenomena. Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining real order or complex order. Fractional calculus is very popular in many fields such as mechanics, dynamics, modelling, mathematical economics, viscoelasticity, biology, electrical engineering, and so on [1-10]. However, the definition of fractional derivative and integral is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-14].

In this paper, based on Jumarie type of R-L fractional calculus, we find the fractional Fourier series expansion of the following two type of α-fractional trigonometric functions:

$$[-r^2+(-r^3+r)cos_\alpha(\theta^\alpha)]\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}, \tag{1}$$

$$[(r^3+r^2)sin_\alpha(\theta^\alpha)]\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}. \tag{2}$$

Where $0<\alpha\leq 1$, and r is any real number. A new multiplication of fractional analytic functions plays an important role in this paper. The main methods we used are fractional Euler's formula and the fractional power series expansion of complex fractional analytic function. In fact, the fractional Fourier series expansion of the above two types of fractional trigonometric functions are generalizations of the Fourier series expansion of trigonometric functions in traditional calculus. On the other hand, some examples are given to illustrate our results.

II. PRELIMINARIES

First, the fractional calculus used in this paper and its properties are introduced below.

Definition 2.1 ([15])**:** If $0<\alpha\leq 1$, and θ_0 is a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{\theta_0}D^{\alpha}_{\theta}\right)[f(\theta)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{\theta_0}^{\theta}\frac{f(t)-f(\theta_0)}{(\theta-t)^\alpha}dt\ . \tag{3}$$

And the Jumarie's modified R-L α-fractional integral is defined by

$$\left({}_{\theta_0}I_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(\alpha)}\int_{\theta_0}^{\theta}\frac{f(t)}{(\theta-t)^{1-\alpha}}dt\ , \tag{4}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([16])**:** *If* $\alpha, \beta, \theta_0, C$ *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{\theta_0}D_{\theta}^{\alpha}\right)\left[(\theta-\theta_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\theta-\theta_0)^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_{\theta_0}D_{\theta}^{\alpha}\right)[C] = 0. \tag{6}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([17])**:** Assume that θ, θ_0, and a_k are real numbers for all k, $\theta_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval containing θ_0, then we say that $f_\alpha(\theta^\alpha)$ is α-fractional analytic at θ_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([18])**:** Let $0 < \alpha \leq 1$, and θ_0 be a real number. If $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions defined on an interval containing θ_0 ,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes k}, \tag{7}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes k}. \tag{8}$$

Then we define

$$\begin{aligned} & f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} \otimes \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(\theta-\theta_0)^{k\alpha}. \end{aligned} \tag{9}$$

Equivalently,

$$\begin{aligned} & f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{10}$$

Definition 2.5 ([19])**:** If $0 < \alpha \leq 1$, and θ is any real number. The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes k}. \tag{11}$$

In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes 2k}, \tag{12}$$

and

$$sin_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^{\alpha}\right)^{\otimes(2k+1)}. \tag{13}$$

Proposition 2.6 (fractional Euler's formula)**:** *Let* $0 < \alpha \leq 1, \theta$ *be a real number, then*

$$E_\alpha(i\theta^\alpha) = cos_\alpha(\theta^\alpha) + isin_\alpha(\theta^\alpha) \,. \tag{14}$$

Definition 2.7: Suppose that $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions. Then $(f_\alpha(\theta^\alpha))^{\otimes n} = f_\alpha(\theta^\alpha) \otimes \cdots \otimes f_\alpha(\theta^\alpha)$ is called the nth power of $f_\alpha(\theta^\alpha)$. On the other hand, if $f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) = 1$, then $g_\alpha(\theta^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $(f_\alpha(\theta^\alpha))^{\otimes -1}$.

Definition 2.8 (fractional Fourier series) ([20])**:** If $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha)$ is a α-fractional analytic function at $\theta = 0$ with the same period T_α of $E_\alpha(i\theta^\alpha)$. Then the α-fractional Fourier series expansion of $f_\alpha(\theta^\alpha)$ is

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos_\alpha(k\theta^\alpha) + b_k sin_\alpha(k\theta^\alpha), \tag{15}$$

where
$$\begin{cases} a_0 = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^{\alpha}\right)[f_\alpha(\theta^\alpha)], \\ a_k = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^{\alpha}\right)[f_\alpha(\theta^\alpha) \otimes cos_\alpha(k\theta^\alpha)], \\ b_k = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^{\alpha}\right)[f_\alpha(\theta^\alpha) \otimes sin_\alpha(k\theta^\alpha)], \end{cases} \tag{16}$$

for all positive integers k.

Definition 2.9: Let $0 < \alpha \leq 1$, $i = \sqrt{-1}$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$, $p_\alpha(\theta^\alpha)$, $q_\alpha(\theta^\alpha)$ be α-fractional real analytic at $\theta = \theta_0$. Let $z_\alpha(\theta^\alpha) = f_\alpha(\theta^\alpha) + ig_\alpha(\theta^\alpha)$ and $w_\alpha(\theta^\alpha) = p_\alpha(\theta^\alpha) + iq_\alpha(\theta^\alpha)$ be complex analytic at $\theta = \theta_0$. Define

$$\begin{aligned} & z_\alpha(\theta^\alpha) \otimes w_\alpha(\theta^\alpha) \\ &= \left(f_\alpha(\theta^\alpha) + i\, g_\alpha(\theta^\alpha)\right) \otimes \left(p_\alpha(\theta^\alpha) + i\, q_\alpha(\theta^\alpha)\right) \\ &= [f_\alpha(\theta^\alpha) \otimes p_\alpha(\theta^\alpha) - g_\alpha(\theta^\alpha) \otimes q_\alpha(\theta^\alpha)] + i[f_\alpha(\theta^\alpha) \otimes q_\alpha(\theta^\alpha) + g_\alpha(\theta^\alpha) \otimes p_\alpha(\theta^\alpha)]. \end{aligned} \tag{17}$$

III. RESULTS AND EXAMPLES

To obtain the main result in this paper, the following lemma is needed.

Lemma 3.1: *Suppose that* $0 < \alpha \leq 1$, *and* $z_\alpha(\theta^\alpha)$ *is a complex* α*-fractional analytic function, then*

$$z_\alpha(\theta^\alpha) \otimes \left[1 - z_\alpha(\theta^\alpha) - \left(z_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes -1} = \sum_{k=1}^{\infty} a_k \left(z_\alpha(\theta^\alpha)\right)^{\otimes k} \,. \tag{18}$$

Where $\{a_k\}_{k=1}^{\infty}$ *is Fibonacci sequence, that is,* $a_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$ *for all positive integers* k.

Proof If $z_\alpha(\theta^\alpha) \otimes \left[1 - z_\alpha(\theta^\alpha) - \left(z_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes -1} = \sum_{k=0}^{\infty} a_k \left(z_\alpha(\theta^\alpha)\right)^{\otimes k}$, then

$$\left[1 - z_\alpha(\theta^\alpha) - \left(z_\alpha(\theta^\alpha)\right)^{\otimes 2}\right] \otimes \sum_{k=0}^{\infty} a_k \left(z_\alpha(\theta^\alpha)\right)^{\otimes k} = z_\alpha(\theta^\alpha) \,. \tag{19}$$

Thus,

$$\sum_{k=0}^{\infty} a_k \left(z_\alpha(\theta^\alpha)\right)^{\otimes k} - \sum_{k=0}^{\infty} a_k \left(z_\alpha(\theta^\alpha)\right)^{\otimes (k+1)} - \sum_{k=0}^{\infty} a_k \left(z_\alpha(\theta^\alpha)\right)^{\otimes (k+2)} = z_\alpha(\theta^\alpha). \tag{20}$$

And hence,

$$a_0 = 0, a_1 = 1, \text{ and } a_{k-2} + a_{k-1} = a_k \text{ for all } k \geq 2. \tag{21}$$

Therefore, $\{a_k\}_{k=1}^{\infty}$ is Fibonacci sequence. Q.e.d.

The following is the major result of this paper.

Theorem 3.2: *If* $0 < \alpha \leq 1$, *and* r *is any real number, then*

$$\begin{aligned} & [-r^2 + (-r^3 + r)cos_\alpha(\theta^\alpha)] \otimes [r^4 + r^2 + 1 + (2r^3 - 2r)cos_\alpha(\theta^\alpha) - 2r^2 cos_\alpha(2\theta^\alpha)]^{\otimes -1} \\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] r^k cos_\alpha(k\theta^\alpha). \end{aligned} \tag{22}$$

And

$$[(r^3+r^2)sin_\alpha(\theta^\alpha)]\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}$$

$$=\sum_{k=1}^{\infty}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right]r^k sin_\alpha(k\theta^\alpha). \quad (23)$$

Proof In Theorem 3.1, let $z_\alpha(\theta^\alpha)=rE_\alpha(i\theta^\alpha)$. Then by fractional Euler's formula, $z_\alpha(\theta^\alpha)=rcos_\alpha(\theta^\alpha)+irsin_\alpha(\theta^\alpha)$. Using Theorem 3.1 yields

$$rE_\alpha(i\theta^\alpha)\otimes\left[1-rE_\alpha(i\theta^\alpha)-\left(rE_\alpha(i\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes-1}=\sum_{k=1}^{\infty}a_k\left(rE_\alpha(i\theta^\alpha)\right)^{\otimes k}, \quad (24)$$

where $\{a_k\}_{k=0}^{\infty}$ is Fibonacci sequence. And hence,

$$\left(rcos_\alpha(\theta^\alpha)+irsin_\alpha(\theta^\alpha)\right)\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)-r^2cos_\alpha(2\theta^\alpha)\right)-i\left(rsin_\alpha(\theta^\alpha)+r^2sin_\alpha(2\theta^\alpha)\right)\right]^{\otimes-1}$$

$$=\sum_{k=1}^{\infty}a_kr^kcos_\alpha(k\theta^\alpha)+i\cdot\sum_{k=1}^{\infty}a_kr^ksin_\alpha(k\theta^\alpha). \quad (25)$$

Thus,

$$\left(rcos_\alpha(\theta^\alpha)+irsin_\alpha(\theta^\alpha)\right)\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)-r^2cos_\alpha(2\theta^\alpha)\right)+i\left(rsin_\alpha(\theta^\alpha)+r^2sin_\alpha(2\theta^\alpha)\right)\right]$$

$$\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)-r^2cos_\alpha(2\theta^\alpha)\right)^{\otimes 2}+\left(rsin_\alpha(\theta^\alpha)+r^2sin_\alpha(2\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes-1}$$

$$=\sum_{k=1}^{\infty}a_kr^kcos_\alpha(k\theta^\alpha)+i\cdot\sum_{k=1}^{\infty}a_kr^ksin_\alpha(k\theta^\alpha). \quad (26)$$

Therefore,

$$\left(rcos_\alpha(\theta^\alpha)+irsin_\alpha(\theta^\alpha)\right)\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)-r^2cos_\alpha(2\theta^\alpha)\right)+i\left(rsin_\alpha(\theta^\alpha)+r^2sin_\alpha(2\theta^\alpha)\right)\right]$$

$$\otimes[1+r^2+r^4+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}$$

$$=\sum_{k=1}^{\infty}a_kr^kcos_\alpha(k\theta^\alpha)+i\cdot\sum_{k=1}^{\infty}a_kr^ksin_\alpha(k\theta^\alpha). \quad (27)$$

So, we obtain

$$rcos_\alpha(\theta^\alpha)\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)-r^2cos_\alpha(2\theta^\alpha)\right)\right]-rsin_\alpha(\theta^\alpha)\otimes[rsin_\alpha(\theta^\alpha)+r^2sin_\alpha(2\theta^\alpha)]$$

$$\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}=\sum_{k=1}^{\infty}a_kr^kcos_\alpha(k\theta^\alpha). \quad (28)$$

And

$$rcos_\alpha(\theta^\alpha)\otimes[rsin_\alpha(\theta^\alpha)+r^2sin_\alpha(2\theta^\alpha)]+rsin_\alpha(\theta^\alpha)\otimes\left[\left(1-rcos_\alpha(\theta^\alpha)-r^2cos_\alpha(2\theta^\alpha)\right)\right]$$

$$\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}=\sum_{k=1}^{\infty}a_kr^ksin_\alpha(k\theta^\alpha). \quad (29)$$

Finally, we get

$$[-r^2+(-r^3+r)cos_\alpha(\theta^\alpha)]\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}$$

$$=\sum_{k=1}^{\infty}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right]r^kcos_\alpha(k\theta^\alpha). \quad (30)$$

And

$$[(r^3+r^2)sin_\alpha(\theta^\alpha)]\otimes[r^4+r^2+1+(2r^3-2r)cos_\alpha(\theta^\alpha)-2r^2cos_\alpha(2\theta^\alpha)]^{\otimes-1}$$

$$=\sum_{k=1}^{\infty}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right]r^ksin_\alpha(k\theta^\alpha). \quad (31)$$

Q.e.d.

Finally, we provide two examples to illustrate our results.

Example 3.3: Let $0 < \alpha \le 1$, and let $r = \frac{2}{3}$ in Theorem 3.2, then we obtain

$$\left[-\frac{4}{9} + \frac{10}{27} cos_\alpha(\theta^\alpha)\right] \otimes \left[\frac{133}{81} - \frac{20}{27} cos_\alpha(\theta^\alpha) - \frac{8}{9} cos_\alpha(2\theta^\alpha)\right]^{\otimes -1}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] \left(\frac{2}{3}\right)^k cos_\alpha(k\theta^\alpha). \quad (32)$$

If $r = -\frac{1}{2}$ in Theorem 3.2, then

$$\left[\frac{1}{8} sin_\alpha(\theta^\alpha)\right] \otimes \left[\frac{21}{16} + \frac{3}{4} cos_\alpha(\theta^\alpha) - \frac{1}{2} cos_\alpha(2\theta^\alpha)\right]^{\otimes -1}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] \left(-\frac{1}{2}\right)^k sin_\alpha(k\theta^\alpha). \quad (33)$$

IV. CONCLUSION

In this paper, we evaluate the fractional Fourier series expansion of two type of fractional trigonometric functions based on Jumarie's modified R-L fractional calculus. The major methods used in this article are fractional Euler's formula and the fractional power series expansion of complex fractional analytic function. Moreover, a new multiplication of fractional analytic functions plays an important role in this article. In fact, the fractional Fourier series expansion of the two types of fractional trigonometric functions are generalizations of the Fourier series expansion of trigonometric functions in ordinary calculus. In the future, we will continue to use these methods to solve the problems in fractional calculus and fractional differential equations.

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